Modelling Between Physics and Mathematics

Thomas Lingefjärd
Email: Thomas.Lingefjard@gu.se
Gothenburg City Council, Sweden

Abstract

In upper secondary school, the relation between mathematics and physics is often such that the topics in mathematics are selected in order to support the teaching of important topics in physics. Both mathematics and physics can benefit from graphical explorations in areas such as displacement and velocity, projectile motion and sound waves. The article suggests a way of using dynamic graphical explorations in GeoGebra as a way to visualize concepts of motion and velocity and velocity. In physics, we also study the example of projectiles and we can let GeoGebra show the building of a graph of a Hammer throw. We can also allow GeoGebra to visualize how noise cancellation earphones works.

1. Introduction

The subject mathematics is important for many other subjects, including physics. The questions we ask in physics often need a mathematical translation to be graphically interpreted and transformed. Mathematics teachers learn physics when interacting with physical questions, questions from physics also give insights and discoveries in mathematical problems. In physics we often ask new questions that benefit from being modelled with mathematical tools. A scientific theory (in physics or in another science) is an isomorphism between physical reality and a mathematical model (see [6].)

The central purpose of teaching is to arrange a situation for someone to learn something from. The central purpose of using graphical explorations or mathematical models for me is that I want students to understand concepts in mathematics or in physics in a better way. I see graphical modelling activities with at least three different purposes:

1. To demonstrate something – a concept, a relation, or a situation
2. To challenge students and engage them in discussions
3. To give students the opportunity to explore some sort of experimental activity over a longer period of time at home or at school.

Students' alternative iconic interpretations of graphical representations are well known from research (se for instance [2]; [4]; [5], [7], [8], [9]). When students are asked to model a velocity–time graph, it often seems as if they are affected by look of the distance–time graph. Students sometimes interpret a representation in the simplest, most literal way possible, such as a bump on a graph corresponding to a hill. This knowledge element is a representational analog of the phenomenological primitives (or p-prim) described by [2]. I strongly recommend that teachers take good time to discuss this representation of a physical object as a point so that students are able to see through the graph (see [5])

Learning from graphical representations require students to understand the characteristics of each representation. The format of a graph includes attributes such as labels, number of axes, and line shapes. Interpretations require finding the slope of lines, minima and maxima, and interceptions ([1], [5]). Students also have to understand which parts of the domain that is represented.)

Example 1.1: Motion

Motion is considered to be a central concept in physics, often viewed through the lenses of mathematics. In Sweden, students meet the concept of motion already in compulsory school, furthermore discussed in the first physics course in upper secondary school. Mathematical tools provide us with graphical resources to study and analyze change of position, velocity or acceleration. The change may be with respect to time or with respect to another quantity. Cartesian coordinate system enables us to model change represented by variables, which vary with respect to each other. Let us consider a person moving from point A to point B. The position of the person varies with respect to time. The displacement will be expressed in a coordinate system with respect to time. Furthermore, the graph over displacement will also offer us some insight in the velocity connected to the person's displacement. Thus motion may be described as physics interacting with a mathematical model involving displacement, velocity, acceleration, and time.

One way to start a lesson in uniform motion is to connect it to something the students are familiar with. Riding a bicycle, travelling in a car or skiing down a mountain slope may be situations which students can easily relate to and may be used to build on their intuitive idea of change in velocity. These experiences are probably dated long before they encounter the concepts of uniform motion in lower or upper secondary school. ([11], 1994) argued that

> The development of images of rate starts with children's image of change in some quantity (e.g., displacement of position, increase in volume), progresses to a loosely coordinated image of two quantities (e.g., displacement of position and duration of displacement), which progresses to an image of the covariation of two quantities so that their measures remain in constant ratio. (p. 128)

Thus, teaching of motion could begin with a discussion about change of displacement or velocity. Changes may be represented graphically as a graph. Nevertheless, to understand the relation between displacement and time or between velocity and time through graphical representation is by no means trivial.

1.1.1 Classroom activity

Let us consider a classroom activity where students are asked to observe and investigate either a skier or a motorcycle in motion. This may be introduced graphically as shown in figure 1a & 1b. The skiers and the motorcycle driver's motion starts at time t = 0.

Figure 1a & 1b. A skier and a motorcycle and its motion illustrated by GeoGebra.

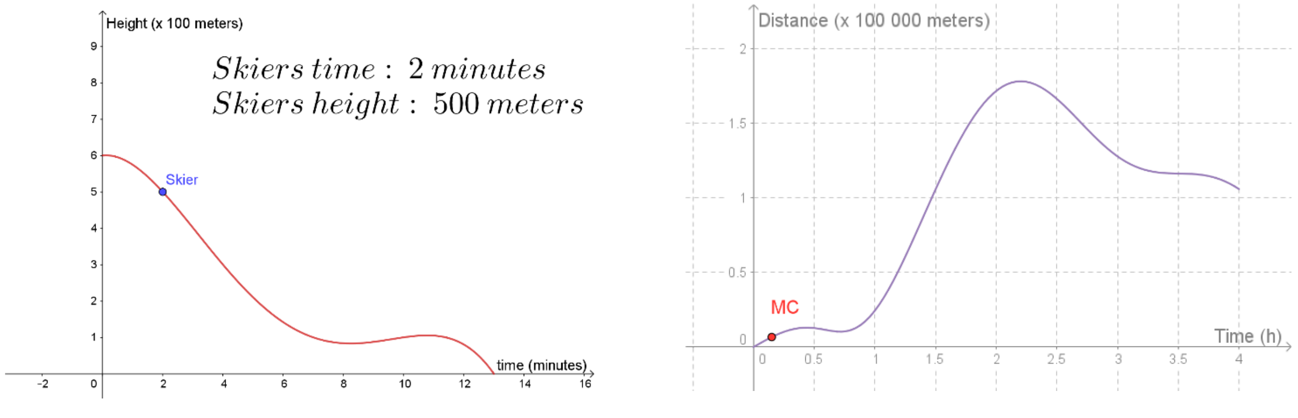


Figure 1a shows the graphical representation of a skier's displacement and figure 1b shows the motorcycle's displacement. You and your students may explore the dynamic version of the figures through the GeoGebra files at https://www.geogebra.org/m/wvgukf3c and at https://www.geogebra.org/m/SVDrYnvQ.

Both of these applets allow you to move the points. Using these applets the teacher may facilitate a discussion by moving the blue point labelled skier (representing the skier) or the red point labelled MC (representing the motorcycle) and ask students to explain what change in displacement means. Students should be encouraged to express their ideas in their own words before a formal explanation is provided.

Usually it is difficult for students to translate the mental image of a skier or a motorcycle into a point on a graph. They should be encouraged to visualize that movement away from the starting point is related to an either a positive velocity upwards or when talking about skiing we might consider the positive movement as downwards. In the motorcycle case it might be the opposite and downwards is related to a negative direction. We measure the displacement of the skier or the motorcycle along the graph and we can ask the students where the skier or the motorcycle has larger or smaller velocity. Since the displacement is not graphed as a straight line, the velocity must be changing over the minutes or the hours.

If we look carefully at the graphical representation in Figure 1a and 1b, we may observe that the skier most likely is going faster in the beginning and then the velocity goes down after 8 minutes and in figure 1 b we see that motorcycle is running rather slowly at first since it only covers a distance of 25 kilometers in the first hour. It also appears as if the motorcycle partly turns around and moves back between 25 minutes to 45 minutes after the start of the journey. Thereafter it changes its course once again.

Teachers may ask students to reason about why skier is skiing like this or why the motorcycle's driver is steering the motorcycle in that way. Let the students suggest probably reasons why the motion is graphed like this. It is naturally to ski the fastest in the beginning and then slow down. Perhaps the motorcycle driver was searching for the correct road. A skier and a motorcycle that is in motion will also have a velocity. How can we view velocity in a distance time graph? What is velocity? Our believe is that students have a general understanding of velocity, perhaps thought of as a measurement of how fast they are able to move from position A to position B. The faster I am moving from position A to position B, the higher is my velocity. At this point of the lesson, it might be suitable to introduce the concept of the slope of the graph and its meaning.

I spent one and a half lesson when talking and discussing about the interpretation of the two graphs above and I am still unsure of all students understood this the way I wanted. I asked them to discuss in pairs and to relate to their own experiences of displacement and velocity and we discussed the "hill problem" several times. A handful student (there was 25 students in the class) were still reluctant to adapt the graphical model of displacement and the possibility to see velocity in the model.

For helping the students to visualize the velocity in some way, it is important to introduce the concept of slope of a curve at a specific point. Is it likely that the skier is skiing in the same velocity the whole time? What do height mean in figure 1a? Is the motorcycle changing its direction at a specific point on the graph in figure 1b? How do we translate that into common behavior when skiing or steering a motorcycle? We can have a rather long discussion with students about where the graphs shows different slope at different time and we could discuss what that means in a physical context. Such an example enables us to develop the concept image ([10]) of slope in the context of graphical representation.

The next lesson could be a more general lesson where we do not care about the origin of the displacement. We just have a displacement of some sort in a uniform motion and the displacement of graphically represented side by side with the velocity. This is a much simpler situation since no skier or motorbike is involved, but at the same time it is a quite more demanding situation since there is a correlated graph of the velocity.

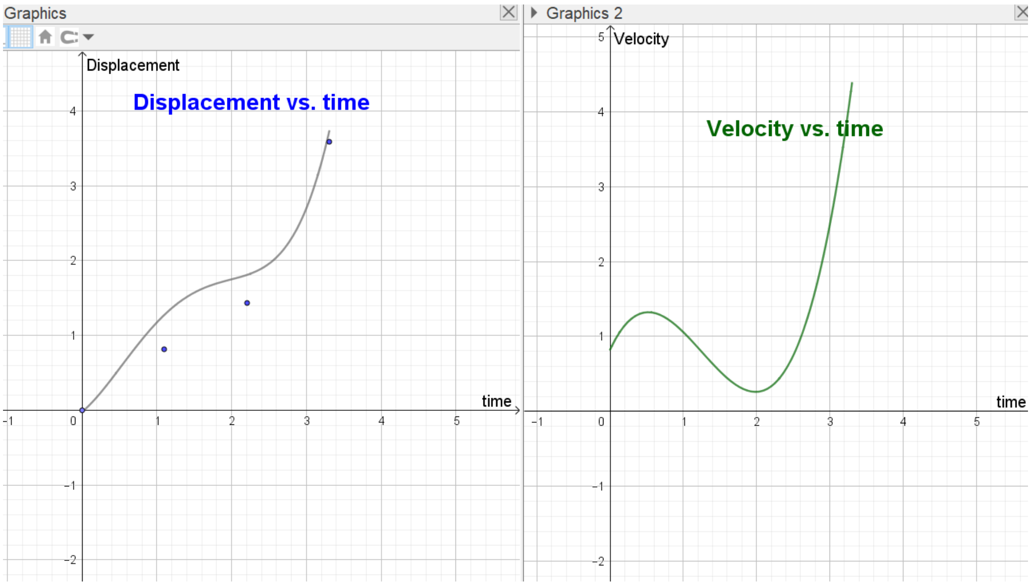


Figure 2: Motion with displacement and velocity illustrated by GeoGebra.

As we see, the position for both displacement and velocity vs. time is adjustable since we are adjusting the displacement over time by moving one or several of the blue points that are defining the curve to the left. If you do that, you will find that the velocity, with a graphical representation in Graphics 2, will also be changed. Since GeoGebra offers two different windows of Graphics this is easy to arrange. You and your students may explore the dynamic version of the figure through the GeoGebra file at https://www.geogebra.org/m/rv87gp3j.

Altogether, I consider it important to take this introduction to displacement and velocity rather slow and I used altogether four lessons including solving different problems from the textbook in physics related to the concepts of displacement and velocity. Perhaps you can do it quicker with your students or perhaps you prefer to do it slower.

1.2 Hammer throw

In physics 2 (studied in year 3 at upper secondary school in the natural science program and in the technical program) students are expected to understand the mathematics needed for dividing a velocity in perpendicular components, trigonometry, the graph of a second degree parabola, and that the velocity in the *x*-direction labelled *Vx* separated from the velocity in the *y*-direction labelled *Vy*.

There are three distinct variables in a Hammer throw:

- The height of the release of the hammer
- The velocity of the release of the hammer
- The angle of the release of the hammer

This GeoGebra file has been used as a Demonstration and as an Experimental activity and as a test item.

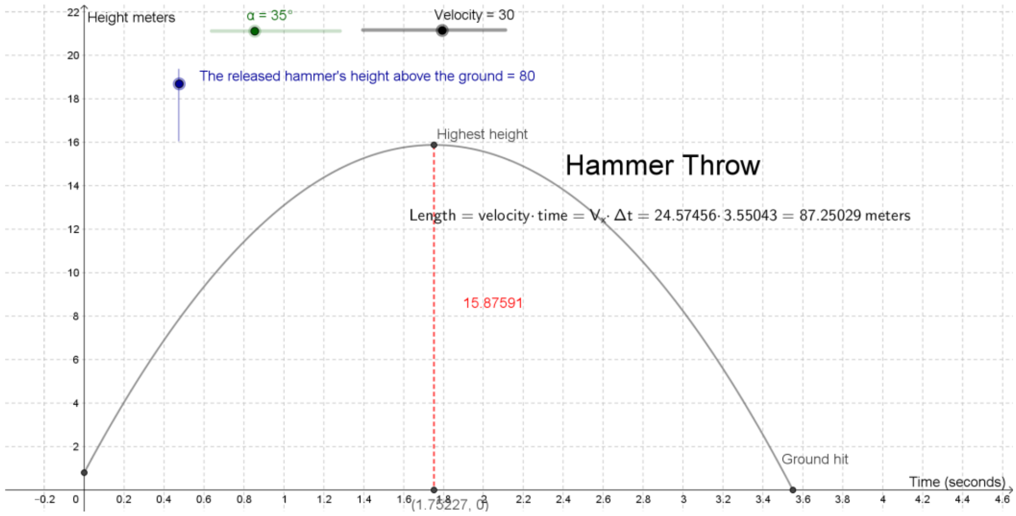


Figure 3: Hammer Throw with different variables illustrated by GeoGebra.

You find the Hammer Throw model at https://www.geogebra.org/m/f3apcvae

1.3 Sound reducing

How do noise cancellation earphones function? The fact is that simple sinus waves can help us understand that. Most music can be expressed as sound waves. A tone as an A may be defined in GeoGebra as $y = \sin(880\pi x)$. See the graph below.

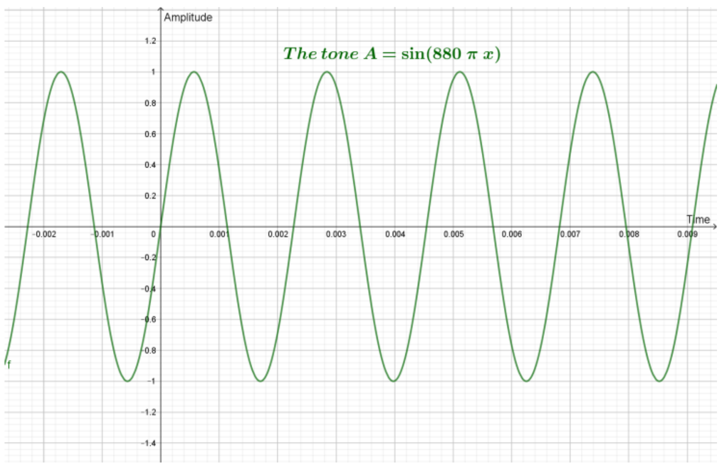


Figure 4: The tone A visualized by GeoGebra.

You find the GeoGebra file at https://www.geogebra.org/m/dm6hjazx

1.3.1 Adding sound waves

When listening to music we normally hear more tones than just one. Can we add sine waves? See below where I have added the tones A, C#, and E. (You might see the similarity with a chord.)

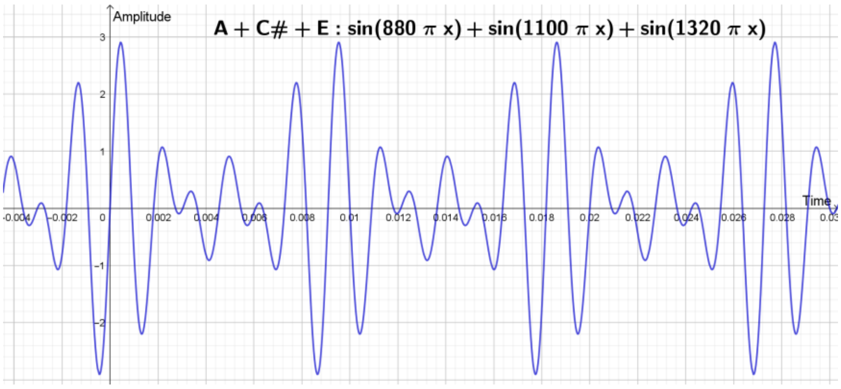


Figure 5: A chord with three tones visualized by GeoGebra.

You find the GeoGebra file at https://www.geogebra.org/m/bthermyd

Even if the graphical representation not looks as a sine wave any more, we can still see a certain pattern in the curve. Moreover, it is still a *periodical* function.

1.3.2 To reduce sound

Noise cancellation works in this way. Your earphones need to monitor the noise that is reaching your ears. When that is done, then your earphones will do regression models over all sound waves in the surrounding sound. Third step is to use the regression models and let the earphones emits a sound wave with the same amplitude but with inverted phase to the original sound. The waves combine to form a new wave, in a process called interference, and effectively cancel each other out in an effect, which is called destructive interference. Your earphones add new sound, and the noise disappear at the same time.

Here is an example. To the left is the tone A from earlier. To the right is a phase shift with another sine curve with the same amplitude and period.

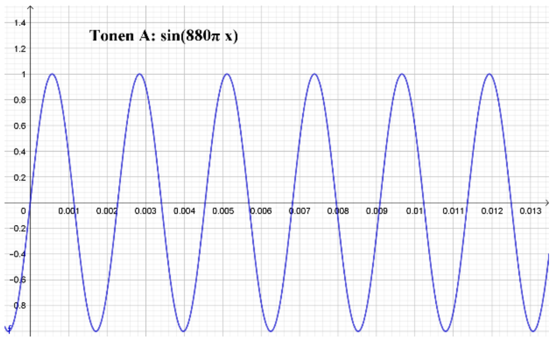


Figure 6: The tone A in GeoGebra.

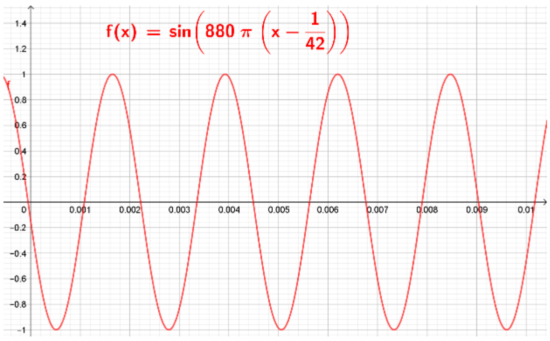
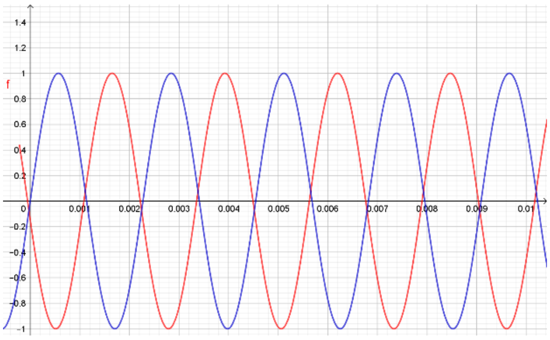


Figure 7: The phase shift to an A in GeoGebra.

In this simple example we are playing that we like to reduce the noise A. We can actually do that with the curve to the right. In the next graph below, we see those two curves in the same coordinate system and thereafter we see them added to each other. We note that the amplitude is reduced to under a fifth of the original sound level. Destructive interference works.



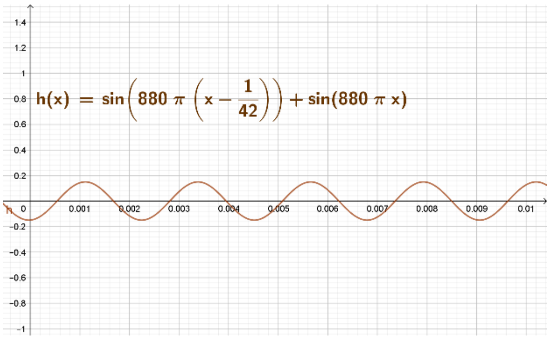


Figure 8: The tone A and an inverted phase. Figure 9: The amplitude is down to a fifth.

You find the GeoGebra file to the model in figure 9 at https://www.geogebra.org/m/brzsqxfm

2. Results

My results are two folded. First, it seemed (although very hard to measure) as if many of the students in the classrooms of physics 1 and physics 2 became interested in the modelling activities. At least half of the students downloaded GeoGebra, played with the applets, and asked questions. Secondly, the students who were in my classes did rather well on the test in physics 1 and in physics 2 compared with general expectations. About 65 % of the students in physics 1 managed to solve two exam questions regarding displacement and velocity. All students in physics 2 managed to solve an extensive exam question about projectile motion.

The overall impression was that the modelling in GeoGebra alerted an interest of physics.

Students' opinions about the GeoGebra models was measured with a questionnaire. The students favored different aspects such as:

a) The possibility to move back and forth with GeoGebra,

b) Possible to test different parameters in the Hammer Throw,

c) Good to investigate the connection between displacement and velocity,

d) It is quite fun to learn mathematics and physics with GeoGebra,

e) I enjoyed learning how my ear phones work

None of the students expressed negative opinions regarding the GeoGebra demonstrations and laboratory activities. It can be that they consider this as a modern and accurate way of teaching.

3. Discussion

I consider the possibility to visualize the isomorphism between physical motion and a mathematical model in GeoGebra as a promising way to help more students to understand how a displacement – time graph is related to a velocity – time graph. The hammer throw programming in GeoGebra proved to be an excellent way to enable students to investigate not just a hammer throw but also to learn about projectiles in physics. Modern technology such as it manifest itself in a noise-cancellation ear phone combines regression analysis from mathematics and wave physics in an interesting mixture. GeoGebra is an excellent tool for visualizing concepts in mathematics and in physics in upper secondary school.

4. References

[1]. Ainsworth, S., Bibby, P., Wood, D. (1997). Information technology and multiple representations: new opportunities - new problems. *Journal of Information for Teacher Education*, 6 (2), pp. 93–104.

[2]. diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition & Instruction*, 10(2/3), 105–226.

[3]. Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103–131.

[4]. Elby, A. (2000).What students' learning of representations tells us about constructivism. *Journal of Mathematical Behavior*, Volume 19, issue 4 (4th quarter, 2000), p. 481-502.

[5]. Friel, S. N., Curcio, F. R., & Bright, G. W. (2001). Making sense of graphs: critical factors influencing comprehension and instructional implications: *Journal for Research in Mathematics Education*, 32, 124- 158.

[6]. Frigg, R. (2006). Scientific Representation and the Semantic View of Theories. *THEORIA. An International Journal for Theory, History and Foundations of Science*, (1), 21, 49-65. ISSN 2171-679X.

[7]. Kohl, B. (2001).Towards an Understanding of how Students Use Representations in Physics Problem Solving. Ph. D. thesis at University of Colorado.

[8]. Leinhardt, G., Zaslavsky, O., & Stein, M. M. (1990). Functions, graphs, and graphing: Tasks, learning and teaching. *Review of Educational Research*, 60(1), 1–64.

[9]. Lingefjärd & Farahani, D. (2017). The Elusive Slope. *International Journal for Science and Mathematics Education*. http://dx.doi.org/10.1007/s10763-017-9811-9

[10]. Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151–169.

[11]. Thompson, P. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies of Mathematics*. *26*(2), 229–274.